Minqi's Solutions to the Exercises and Problems of Michael Artin (2011), Algebra (2nd Edition)

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Contents

CHAPTER 1

Matrices

1.1. Exercise 1.1: The Row Index and the Column Index

What are the entries a_{21} and a_{23} of the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 7 & 8 \\ 0 & 9 & 4 \end{bmatrix}$?

Answer. 2 and 8. □

1.2. Exercise 1.2: Matrix Multiplication is not Commutative

Determine the products AB and BA for the following values of A and B:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} -8 & -4 \\ 9 & 5 \\ -3 & -2 \end{bmatrix}; A = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix}.$$

Answer. For the 1st pair,

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, BA = \begin{bmatrix} -20 & -28 & -28 \\ 24 & 33 & 32 \\ -9 & -12 & -11 \end{bmatrix}.$$

For the 2nd pair,

$$AB = \begin{bmatrix} 18 & 4 \\ 12 & 0 \end{bmatrix}, BA = \begin{bmatrix} 2 & 16 \\ 5 & 16 \end{bmatrix}.$$

□

1.3. Exercise 1.3: Row×Column and Column×Row

Let $A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$ be a row vector, and let $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ be a column vector.

Compute the products AB and BA.

Answer.

$$AB = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

$$BA = \begin{bmatrix} b_1a_1 & b_1a_2 & \dots & b_1a_n \\ b_2a_1 & b_2a_2 & \dots & b_2a_n \\ \vdots & & & \\ b_na_1 & b_na_2 & \dots & b_na_n \end{bmatrix}.$$

□

1.4. Exercise 1.4: Matrix Multiplication is Associative

Verify the associative law for the matrix product $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

Answer.

$$\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 8 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 38 \\ 14 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 14 \end{bmatrix} = \begin{bmatrix} 38 \\ 14 \end{bmatrix}$$

□

1.5. Exercise 1.5: The Number of Multiplications Required to Multiply 3 Matrices

Let A, B and C be matrices of size $l \times m$, $m \times n$ and $n \times p$. How many multiplications are required to compute the product AB? In which order should the triple product ABC be computed, so as to minimize the number of multiplications required?

Answer. AB has $l \times n$ entries, each in need of m multiplications. Therefore $l \times m \times n$ multiplications are required to compute the product AB.

There are 2 ways to compute ABC:

(1) Compute $(AB)C$ with $l \times m \times n + l \times n \times p$ multiplications;
(2) Compute $A(BC)$ with $m \times n \times p + l \times m \times p$ multiplications.

Therefore,

(1) If $lmn + lnp > mnp + lmp$, the product ABC should be computed in the order of $A(BC)$.
(2) If $lmn + lnp < mnp + lmp$, the product ABC should be computed in the order of $(AB)C$.
(3) Otherwise, both orders are equally good.

□

1.6. Exercise 1.6: 2×2 Upper Triangular Matrices

Compute $\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}^n$.

Answer.

$$\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & na \\ & 1 \end{bmatrix}.$$

□

1.7. Exercise 1.7: 3×3 Upper Triangular Matrices

Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

PROOF. We claim that,

$$\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}.$$

Set $n = 2$ and we can verify it by showing that,

$$\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 & 3 \\ & 1 & 2 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & \frac{2*(2+1)}{2} \\ & 1 & 2 \\ & & 1 \end{bmatrix}.$$

Now suppose that there exists some $n_0 \in \mathbb{N}, n_0 \geq 2$ such that,

$$\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{n_0} = \begin{bmatrix} 1 & n_0 & \frac{n_0(n_0+1)}{2} \\ & 1 & n_0 \\ & & 1 \end{bmatrix}.$$

It follows that,

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{n_0+1} &= \begin{bmatrix} 1 & n_0 & \frac{n_0(n_0+1)}{2} \\ & 1 & n_0 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & n_0+1 & \frac{n_0(n_0+1)}{2} + n_0 + 1 \\ & 1 & n_0+1 \\ & & 1 \end{bmatrix}. \end{aligned}$$

Note that,

$$\begin{aligned} \frac{n_0(n_0+1)}{2} + n_0 + 1 &= \frac{n_0(n_0+1)}{2} + \frac{2n_0}{2} + \frac{2}{2} \\ &= \frac{n_0(n_0+1) + 2n_0 + 2}{2} \\ &= \frac{n_0(n_0+1) + 2(n_0+1)}{2} \\ &= \frac{(n_0+2)(n_0+1)}{2} \\ &= \frac{(n_0+1)((n_0+1)+1)}{2}. \end{aligned}$$

Therefore the equation holds for all $n \in \mathbb{N}, n \geq 2$. □

1.8. Exercise 1.8: Block Multiplications

Compute the following products by block multiplications:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right], \quad \left[\begin{array}{c|cc} 0 & 1 & 2 \\ \hline 0 & 1 & 0 \\ 3 & 0 & 1 \end{array}\right] \left[\begin{array}{c|cc} 1 & 2 & 3 \\ \hline 4 & 2 & 3 \\ 5 & 0 & 4 \end{array}\right]$$

ANSWER.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 2 & 8 & 6 & 17 \\ 0 & 2 & 1 & 4 \\ \hline 1 & 3 & 2 & 3 \\ 1 & 1 & 0 & 2 \end{array}\right].$$

$$\left[\begin{array}{c|cc} 0 & 1 & 2 \\ \hline 0 & 1 & 0 \\ 3 & 0 & 1 \end{array}\right]\left[\begin{array}{c|cc} 1 & 2 & 3 \\ \hline 4 & 2 & 3 \\ 5 & 0 & 4 \end{array}\right]$$

$$= \left[\begin{array}{c|c} 0 * 1 + \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} & 0 * \begin{pmatrix} 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \\ \hline \begin{pmatrix} 0 \\ 3 \end{pmatrix} * 1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 4 \\ 5 \end{pmatrix} & \begin{pmatrix} 0 \\ 3 \end{pmatrix} * \begin{pmatrix} 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|cc} 14 & 2 & 11 \\ \hline 4 & 2 & 3 \\ 8 & 6 & 13 \end{array}\right].$$

□

1.9. Exercise 1.9: Laws of Matrix Operations

Let A, B be square matrices.

(a) When is $(A+B)(A-B) = A^2 - B^2$?
(b) Expand $(A+B)^3$.

ANSWER. (a) When $BA = AB$.
(b) $(A+B)^3 = A^3 + ABA + BAA + BBA + AAB + ABB + BAB + B^3$.

□

1.10. Exercise 1.10: Multiplications with Diagonal Matrices

Let D be the diagonal matrix with diagonal entries $d_1, \dots, d_n$, and let $A = (a_{ij})$ be an arbitrary $n \times n$ matrix. Compute the products DA and AD.

ANSWER.

$$DA = \begin{bmatrix} d_1a_{11} & d_1a_{12} & \dots & d_1a_{1n} \\ d_2a_{21} & d_2a_{22} & \dots & d_2a_{2n} \\ \dots & \dots & \dots & \dots \\ d_na_{n1} & d_na_{n2} & \dots & d_na_{nn} \end{bmatrix},$$

$$AD = \begin{bmatrix} d_1a_{11} & d_2a_{12} & \dots & d_na_{1n} \\ d_1a_{21} & d_2a_{22} & \dots & d_na_{2n} \\ \dots & \dots & \dots & \dots \\ d_1a_{n1} & d_2a_{n2} & \dots & d_na_{nn} \end{bmatrix}.$$

□