Minqi's Solutions to the Exercises of Sheldon Axler (2014), Linear Algebra Done Right (3rd Edition)

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Contents

CHAPTER 1

Vector Spaces

1.1. Exercise 1.A.1: Multiplicative Inverse of Complex Numbers

Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$1/(a+bi) = c+di$$

Answer. Since at least one of the real numbers a, b is different from 0, we have $a^2+b^2 \neq 0$. We can then let

$$c = \frac{a}{a^2+b^2},$$

$$d = \frac{-b}{a^2+b^2}.$$

Then

$$\begin{aligned}
&(a+bi)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right) \\
&= \left(a\frac{a}{a^2+b^2} - b\left(-\frac{b}{a^2+b^2}\right)\right) + \left(a\left(-\frac{b}{a^2+b^2}\right) + b\frac{a}{a^2+b^2}\right)i \\
&= 1+0i \\
&= 1.
\end{aligned}$$

□

1.2. Exercise 1.A.2: A Cubic Root of 1

Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Answer.

$$\begin{aligned}
&\frac{-1+\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2} \\
&= \frac{-2-2\sqrt{3}i}{4} \cdot \frac{-1+\sqrt{3}i}{2} \\
&= \frac{(-2)(-1) - (-2\sqrt{3})(\sqrt{3}) + (-2\sqrt{3} + (-2\sqrt{3})(-1))i}{8} \\
&= 1
\end{aligned}$$

□

1.3. Exercise 1.A.3: Square Roots of i

Find two distinct square roots of i.

ANSWER. Note that

$$i = e^{i\frac{\pi}{2}}.$$

Thus the following two numbers are two distinct square roots of i:

$$\pm e^{i\frac{\pi}{4}} = \pm(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i).$$

□

1.4. Exercise 1.A.4: Commutativity of Addition of Complex Numbers

Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

PROOF. Suppose $\alpha = a + bi$ and $\beta = c + di$, where $a, b, c, d \in \mathbb{R}$. Then the definition of addition of complex numbers shows that

$$\alpha + \beta = (a + c) + (b + d)i$$

and

$$\beta + \alpha = (c + a) + (d + b)i.$$

The equations above and the commutativity of addition of real numbers show that $\alpha + \beta = \beta + \alpha$. □

1.5. Exercise 1.A.5: Associativity of Addition of Complex Numbers

Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

PROOF. Suppose $\alpha = a+bi$, $\beta = c+di$ and $\lambda = e+fi$, where $a, b, c, d, e, f \in \mathbb{R}$. Then the definition of addition of complex numbers shows that

$$\begin{aligned}(\alpha + \beta) + \lambda &= ((a + c) + (b + d)i) + e + fi \\ &= ((a + c) + e) + ((b + d) + f)i\end{aligned}$$

and

$$\begin{aligned}\alpha + (\beta + \lambda) &= a + bi + ((c + e) + (d + f)i) \\ &= (a + (c + e)) + (b + (d + f))i.\end{aligned}$$

The equations above and the associativity of addition of real numbers show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$. □

1.6. Exercise 1.A.6: Associativity of Multiplication of Complex Numbers

Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

PROOF. Suppose $\alpha = a+bi$, $\beta = c+di$ and $\lambda = e+fi$, where $a, b, c, d, e, f \in \mathbb{R}$. Then the definition of multiplication of complex numbers shows that

$$\begin{aligned}(\alpha\beta)\lambda &= ((ac - bd) + (ad + bc)i)(e + fi) \\ &= ((ac - bd)e - (ad + bc)f) + ((ac - bd)f + (ad + bc)e)i \\ &= (ace - bde - adf - bcf) + (acf - bdf + ade + bce)i\end{aligned}$$

and

$$\begin{aligned}\alpha(\beta\lambda) &= (a+bi)((ce-df)+(cf+de)i)\\ &= (a(ce-df)-b(cf+de))+(a(cf+de)+b(ce-df))i\\ &= (ace-adf-bcf-bde)+(acf+ade+bce-bdf)i.\end{aligned}$$

The equations above and the commutativity of addition of real numbers show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$. □

1.7. Exercise 1.A.7: Existence and Uniqueness of Additive Inverse in $\mathbb{C}$

Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Proof. Suppose $\alpha = a+bi$ where $a, b \in \mathbb{R}$, then $\beta = -a-bi$ satisfies $\alpha+\beta = 0$. Suppose that there exists another $\beta' \in \mathbb{C}$ such that $\alpha + \beta' = 0$. Then

$$\begin{aligned}\beta &= \beta + (\alpha + \beta')\\ &= (\beta + \alpha) + \beta'\\ &= \beta'.\end{aligned}$$

The equations above show that $\beta' = \beta$. □

1.8. Exercise 1.A.8: Existence and Uniqueness of Multiplicative Inverse in $\mathbb{C}$

Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Proof. The existence of β has been shown in Exercise 1.A.1.
Suppose that there exists another $\beta' \in \mathbb{C}$ such that and $\alpha\beta' = 1$. Then

$$\begin{aligned}\beta &= \beta(\alpha\beta')\\ &= (\beta\alpha)\beta'\\ &= \beta'.\end{aligned}$$

The equations above show that $\beta' = \beta$. □

1.9. Exercise 1.A.9: The Distributive Property of Complex Numbers

Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.

Proof. Suppose $\alpha = a+bi$, $\beta = c+di$ and $\lambda = e+fi$, where $a, b, c, d, e, f \in \mathbb{R}$. Then the definition of multiplication and addition of complex numbers shows that

$$\begin{aligned}\lambda(\alpha+\beta) &= (e+fi)((a+c)+(b+d)i)\\ &= (e(a+c)-f(b+d))+(e(b+d)+f(a+c))i\\ &= (ea+ec-fb-fd)+(eb+ed+fa+fc)i\end{aligned}$$

and

$$\begin{aligned}\lambda\alpha+\lambda\beta &= (e+fi)(a+bi)+(e+fi)(c+di)\\ &= ((ea-fb)+(eb+fa)i)+((ec-fd)+(ed+fc)i)\\ &= (ea-fb+ec-fd)+(eb+fa+ed+fc)i.\end{aligned}$$

The equations above and the commutativity of addition of real numbers show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$. □

1.10. Exercise 1.A.10: A Vector Addition and Scalar Multiplication Example in $\mathbb{R}^4$

Find $x \in \mathbb{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Answer.

$$x = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right).$$

□

1.11. Exercise 1.A.11: A Scalar Multiplication Example in $\mathbb{C}^3$

Explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Proof. If there exists such λ then the definition of scalar multiplication in $\mathbb{C}^3$ shows that,

$$\lambda(2 - 3i) = 12 - 5i.$$

Thus

$$\lambda = \frac{12 - 5i}{2 - 3i} = 3 + 2i.$$

But

$$\begin{aligned} \lambda(2 - 3i, 5 + 4i, -6 + 7i) &= (3 + 2i)(2 - 3i, 5 + 4i, -6 + 7i) \\ &= (12 - 5i, 7 + 22i, -32 + 9i) \\ &\neq (12 - 5i, 7 + 22i, -32 - 9i). \end{aligned}$$

Therefore λ does not exist. □

1.12. Exercise 1.A.12: Associativity of Vector Addition in $\mathbb{F}^n$

Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}^n$.

Proof. Suppose $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and $z = (z_1, \dots, z_n)$ where $x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n \in \mathbb{F}$. Then the definition of addition in $\mathbb{F}^n$ shows that

$$\begin{aligned} (x + y) + z &= (x_1 + y_1, \dots, x_n + y_n) + (z_1, \dots, z_n) \\ &= ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n) \end{aligned}$$

and

$$\begin{aligned} x + (y + z) &= (x_1, \dots, x_n) + (y_1 + z_1, \dots, y_n + z_n) \\ &= (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)) \end{aligned}$$

The equations above and the associativity of addition in $\mathbb{F}$ show that $(x + y) + z = x + (y + z)$. □

1.13. Exercise 1.A.13: Associativity of Scalar Multiplication in $\mathbb{F}^n$

Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and all $a, b \in \mathbb{F}$.

PROOF. Suppose $x = (x_1, \dots, x_n)$ where $x_1, \dots, x_n \in \mathbb{F}$. Then the definition of scalar multiplication in $\mathbb{F}^n$ shows that

$$(ab)x = ((ab)x_1, \dots, (ab)x_n)$$

and

$$\begin{aligned} a(bx) &= a(bx_1, \dots, bx_n) \\ &= (a(bx_1), \dots, a(x_n)) \end{aligned}$$

The equations above and the associativity of multiplication in $\mathbb{F}$ show that $(ab)x = a(bx)$. □

1.14. Exercise 1.A.14: Scalar Multiplication by 1 in $\mathbb{F}^n$

Show that $1x = x$ for all $x \in \mathbb{F}^n$.

PROOF. Suppose $x = (x_1, \dots, x_n)$ where $x_1, \dots, x_n \in \mathbb{F}$. Then the definition of scalar multiplication in $\mathbb{F}^n$ and the fact that 1 is the multiplicative identity of $\mathbb{F}$ shows that

$$\begin{aligned} 1x &= (1x_1, \dots, 1x_n) \\ &= (x_1, \dots, x_n) \\ &= x. \end{aligned}$$

□

1.15. Exercise 1.A.15: Left-distributive Property of Operations in $\mathbb{F}^n$

Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and $x, y \in \mathbb{F}^n$.

PROOF. Suppose $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ where $x_1, \dots, x_n \in \mathbb{F}$ and $y_1, \dots, y_n \in \mathbb{F}$. Then the definition of addition and scalar multiplication in $\mathbb{F}^n$ shows that

$$\begin{aligned} \lambda(x + y) &= \lambda(x_1 + y_1, \dots, x_n + y_n) \\ &= (\lambda(x_1 + y_1), \dots, \lambda(x_n + y_n)) \end{aligned}$$

and

$$\begin{aligned} \lambda x + \lambda y &= (\lambda x_1, \dots, \lambda x_n) + (\lambda y_1, \dots, \lambda y_n) \\ &= (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n) \end{aligned}$$

The equations above and the distributive property of $\mathbb{F}$ show that $\lambda(x + y) = \lambda x + \lambda y$. □

1.16. Exercise 1.A.16: Right-distributive Property of Operations in $\mathbb{F}^n$

Show that $(a+b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and all $x \in \mathbb{F}^n$.

PROOF. Suppose $x = (x_1, \ldots, x_n)$ where $x_1, \ldots, x_n \in \mathbb{F}$. Then the definition of scalar multiplication and addition in $\mathbb{F}^n$ shows that

$$(a+b)x = ((a+b)x_1, \ldots, (a+b)x_n)$$

and

$$\begin{aligned} ax + bx &= (ax_1, \ldots, ax_n) + (bx_1, \ldots, bx_n) \\ &= (ax_1 + bx_1, \ldots, ax_n + bx_n). \end{aligned}$$

The equations above and the distributive property of $\mathbb{F}$ show that $(a+b)x = ax + bx$. □

1.17. Exercise 1.B.1: $-(-v) = v$ in Vector Spaces

Prove that $-(-v) = v$ for every $v \in V$.

PROOF.

$$\begin{aligned} -(-v) &= -(-v) + 0 \\ &= -(-v) + ((-v) + v) \\ &= (-(-v) + (-v)) + v \\ &= 0 + v \\ &= v \end{aligned}$$

□

1.18. Exercise 1.B.2: $av = 0$ Implies $a = 0$ or $v = 0$ in Vector Spaces

Suppose $a \in \mathbb{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

PROOF. For the purpose of arriving at a contradiction, assume that both $a \neq 0$ and $v \neq 0$. Then

$$v = 1v = (\frac{1}{a}a)v = \frac{1}{a}(av) = 0$$

which contradicts with $v \neq 0$.

Therefore $a = 0$ or $v = 0$. □

1.19. Exercise 1.B.3: Existence and Uniqueness of the Solution of a Vector Equation

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

PROOF. We claim that the following $x \in V$ satisfies $v + 3x = w$:

$$x = \frac{1}{3}(w - v).$$

We can verify it by showing that,

$$\begin{aligned} v + 3x &= v + 3 \times \frac{1}{3}(w - v) \\ &= v + w - v \\ &= w. \end{aligned}$$

Suppose that there exists another $x' \in V$ that also satisfies $v + 3x' = w$, then

$$\begin{aligned} 0 &= w - w \\ &= (v + 3x) - (v + 3x') \\ &= v + 3x - v - 3x' \\ &= 3x - 3x'. \end{aligned}$$

Thus

$$\begin{aligned} x' &= x' + \frac{1}{3} \times 0 \\ &= x' + \frac{1}{3}(3x - 3x') \\ &= x' + x - x' \\ &= x. \end{aligned}$$

□

1.20. Exercise 1.B.4: The Empty Set is NOT a Vector Space

The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Answer. The empty set fails to satisfy the "Additive Identity" requirement. Because there does not exist an element $0 \in \varnothing$. □

1.21. Exercise 1.B.5: An Alternative Definition of Vector Space

Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0, \forall v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Proof. We will show that the new condition is equal to the original condition, which states that for every $v \in V$, there exists $w \in V$ such that $v + w = 0$. That the old condition implies the new condition has been shown in Theorem 1.29 "The number 0 times a vector". Therefore it suffices to show that the new condition implies the old condition.

Suppose that the new condition holds.

Pick $v \in V$, and let $w = (-1)v$. It follows that

$$\begin{aligned} v + w &= 1v + (-1)v \\ &= (1 - 1)v \\ &= 0v \\ &= 0. \end{aligned}$$

Therefore the old condition holds. □

1.22. Exercise 1.B.6: A Definition of $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \quad t + (-\infty) = (-\infty) + t = (-\infty),$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = (-\infty), \quad \infty + (-\infty) = 0.$$

Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$? Explain.

ANSWER. No. Because the associativity of vector addition fails:

$$(1024 + \infty) - \infty = \infty - \infty = 0$$

but

$$1024 + (\infty - \infty) = 1024 + 0 = 1024.$$

□

1.23. Exercise 1.C.1: Examples and Counterexamples of Subspaces of $\mathbb{F}^3$

For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$:

(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;
(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;
(c) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\}$;
(d) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$.

ANSWER. (a) Denote $A = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$. We claim that A is a subspace of $\mathbb{F}^3$ with the following verifications.

(i) $(0, 0, 0) \in A$ because $0 + 2 * 0 + 3 * 0 = 0$.

(ii) Pick $u, w \in A$, where $u = (u_1, u_2, u_3), w = (w_1, w_2, w_3)$. It follows that $u + w = (u_1 + w_1, u_2 + w_2, u_3 + w_3)$. We can show that $u + w \in A$ by the following equations:

$$\begin{aligned} &(u_1 + w_1) + 2(u_2 + w_2) + 3(u_3 + w_3) \\ &= (u_1 + 2u_2 + 3u_3) + (w_1 + 2w_2 + 3w_3) \\ &= 0 + 0 = 0. \end{aligned}$$

(iii) Pick $a \in \mathbb{F}$ and $u \in A$, where $u = (u_1, u_2, u_3)$. It follows that $au = (au_1, au_2, au_3)$. We can show that $au \in A$ by the following equations:

$$\begin{aligned} (au_1) + 2(au_2) + 3(au_3) &= a(u_1 + 2u_2 + 3u_3) \\ &= a0 = 0. \end{aligned}$$

(b) Denote $B = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$. B is not a subspace of $\mathbb{F}^3$, because vector $(0, 0, 0) \notin B$ (as $0 + 2 * 0 + 3 * 0 \neq 4$).

(c) Denote $C = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1x_2x_3 = 0\}$. C is not a subspace of $\mathbb{F}^3$ because by setting $u = (0, 1, 1) \in C$ and $w = (1, 1, 0) \in C$, we find out that $u + w = (1, 2, 1) \notin C$.

(d) Denote $D = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$. We claim that D is a subspace of $\mathbb{F}^3$ with the following verifications.

(i) $(0, 0, 0) \in A$ because $0 = 5 * 0$.

(ii) Pick $u, w \in A$, where $u = (u_1, u_2, u_3), w = (w_1, w_2, w_3)$. It follows that $u+w = (u_1+w_1, u_2+w_2, u_3+w_3)$. We can show that $u+w \in A$ by the following equations:

$$\begin{aligned} u_1 + w_1 &= 5u_3 + 5w_3 \\ &= 5(u_1 + w_3) \end{aligned}$$

(iii) Pick $a \in \mathbb{F}$ and $u \in A$, where $u = (u_1, u_2, u_3)$. It follows that $au = (au_1, au_2, au_3)$. We can show that $au \in A$ by the following equations:

$$\begin{aligned} au_1 &= a(5u_3) \\ &= 5(au_3). \end{aligned}$$

□

1.24. Exercise 1.C.2: Supplying Proofs for Examples of Vector Spaces

Verify all the assertions in Example 1.35.

(a) If $b \in \mathbb{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.

(b) The set of continuous real-values functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

(c) The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$

(d) The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}^\infty$.

ANSWER. (a) Denote $A = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$.

If $b = 0$, then A is a subspace by a verification similar to Exercise 1.C.2 (d).

On the other hand, if A is a subspace, then $0 \in A$. It follows that

$$0 = 5 * 0 + b = b.$$

(b) Denote $B = \{f \in \mathbb{R}^{[0,1]} : f \text{ is continuous}\}$.

We can verify that $0 \in B$ because the constant function $f = 0$ is continuous.

Suppose that $u, w \in B$, we can verify that $(u+w) \in B$ because the addition of two complex continuous functions is continuous by Theorem 4.9 of [**Rud76**].

Suppose that $a \in \mathbb{F}$ and $u \in B$. Note that the constant function $f = a$ is continuous. We can verify that $au \in B$ because the multiplication of two continuous complex functions is continuous by Theorem 4.9 of [**Rud76**].

(c) Denote $C = \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is differentiable}\}$.

We can verify that $0 \in C$ because the constant function $f = 0$ is differentiable.

Suppose that $u, w \in C$, we can verify that $(u+w) \in C$ because the addition of two differentiable real functions is differentiable by Theorem 5.3 of [**Rud76**].

Suppose that $a \in \mathbb{R}$ and $u \in C$. Note that the constant function $a(x) = a$ is differentiable. We can verify that $au \in C$ because the multiplication of two differentiable real functions is differentiable by Theorem 5.3 of [**Rud76**].

(d) Denote $D = \{f \in \mathbb{R}^{(0,3)} : f \text{ is differentiable and } f'(2) = b\}$.

Suppose that $b = 0$. We claim that D is a subspace of $\mathbb{R}^{(0,3)}$ with the following verifications.

(i) $0 \in D$ because the constant function $0(x) = 0$ is differentiable and $0'(2) = 0$.

(ii) Suppose that $u, w \in D$. $(u + w)$ is also differentiable by Theorem 5.3 of [**Rud76**]. Also by Theorem 5.3 of [**Rud76**], $(u + v)'(2) = u'(2) + v'(2) = 0 + 0 = 0$. Thus $(u + w) \in D$.

(iii) Suppose that $a \in \mathbb{R}$ and $u \in D$. Note that the constant function $a(x) = a$ is differentiable, so au is differentiable by Theorem 5.3 of [**Rud76**]. Also by Theorem 5.3 of [**Rud76**], $(au)'(2) = a'(2)u(2) + a(2)u'(2) = 0 \times u(2) + a \times 0 = 0$. Thus $au \in D$.

On the other hand, suppose that D is a subspace. Then it implies that $0 \in D$ where $0(x) = 0$, and $0'(x) = 0$ implies that $0'(2) = b = 0$.

(e) Denote $E = \{z_n \in \mathbb{C}^\infty : \lim_{n\to\infty} z_n = 0\}$.

We can verify that $0_n \in E$ where $0_n = (0, 0, 0, \dots)$ because

$$\lim_{n\to\infty} 0_n = 0.$$

Suppose that $u_n, w_n \in E$, we can verify that $(u_n + w_n) \in C$ because $\lim_{n\to\infty}(u_n + w_n) = \lim_{n\to\infty} u_n + \lim_{n\to\infty} w_n = 0 + 0 = 0$ by Theorem 3.3 of [**Rud76**].

Suppose that $a \in \mathbb{F}$ and $u_n \in E$, we can verify that $au_n \in E$ because $\lim_{n\to\infty} au_n = a \lim_{n\to\infty} u_n = a0 = 0$ by Theorem 3.3 of [**Rud76**].

□

1.25. Exercise 1.C.3: A Subspace of $\mathbb{R}^{(-4,4)}$

Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

PROOF. Denote $X = \{f \in \mathbb{R}^{(-4,4)} : f \text{ is differentiable and } f'(-1) = 3f(2)\}$

(1) We claim that $0 \in X$ where $0(x) = 0$. Note that 0 is differentiable and $0'(-1) = 0, 0(2) = 0$, thus $0'(-1) = 3 \times 0(2)$.

(2) Pick $f, g \in X$. That $f + g$ is differentiable follows from Theorem 5.3 of [**Rud76**]. Also by Theorem 5.3 of [**Rud76**], $(f + g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f(2) + g(2)) = 3((f + g)(2))$. Thus $f + g \in X$.

(3) Pick $f \in X$ and $a \in \mathbb{F}$. That af is differentiable follows from Theorem 5.3 of [**Rud76**] by considering a as a constant function $a(x) = a$. Also by

Theorem 5.3 of [**Rud76**], $(af)'(-1) = a(f'(-1)) = a(3f(2)) = 3(af(2)) = 3((af)(2))$. Thus $af \in X$.

□

1.26. Exercise 1.C.4: A Subspace of $\mathbb{R}^{[0,1]}$

Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Proof. Denote $X = \{f \in \mathbb{R}^{[0,1]}$:f is continuous and $\int_0^1 f = b\}$.
Suppose that $b = 0$. We now verify that X is a subspace.

(1) We can verify that $0 \in X$ because $0(x) = 0$ is continuous and $\int_0^1 0(x)dx = 0$.
(2) Pick $f, g \in X$. It follows that $f + g$ is continuous by Theorem 4.9 of [**Rud76**] and Riemann-integrable by Theorem 6.8 of [**Rud76**]. Further by Theorem 6.12 of [**Rud76**], we have $\int_0^1 (f(x) + g(x))dx = \int_0^1 f(x)dx + \int_0^1 g(x) = 0 + 0 = 0$.
(3) Pick $a \in \mathbb{F}, f \in X$. By Theorem 4.9 of [**Rud76**], af is continuous. Thus af is Riemann-integrable by Theorem 6.8 of [**Rud76**]. Further by Theorem 6.12 of [**Rud76**], we have $\int_0^1 (af(x))dx = a\int_0^1 f(x)dx = a \times 0 = 0$.

On the other hand, suppose that X is a subspace. Then it follows that $0 \in X$, which implies that $\int_0^1 0(x)dx = 0 = b$. □

1.27. Exercise 1.C.5: $\mathbb{R}^2$ is NOT a Subspace of $\mathbb{C}^2$

Is $\mathbb{R}^2$ a subspace of the complex vector space $\mathbb{C}^2$?

Answer. No. Because multiplying the vector $(1,1) \in \mathbb{R}^2$ by scalar i gives (i,i) which is not in $\mathbb{R}^2$. □

1.28. Exercise 1.C.6: Comparing Subspaces of $\mathbb{R}^3$ and $\mathbb{C}^3$

(i) Is $\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?
(ii) Is $\{(a,b,c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Answer. (i) Yes. Let's observe the real function $f(x) = x^3$. $f(x)$ is a 1-1 correspondence between $\mathbb{R}$ and $\mathbb{R}$. Therefore the condition $a^3 = b^3$ is equivalent to $a = b$.
(ii) No. Denote $X = \{(a,b,c) \in \mathbb{C}^3 : a^3 = b^3\}$. Set $u = (2, 2e^{\frac{2\pi}{3}i}, 0)$ and $v = (e^{\frac{2\pi}{3}i}, 1, 0)$, then $u+v = (2+e^{\frac{2\pi}{3}i}, 2e^{\frac{2\pi}{3}i}+1, 0)$. Note that $u \in X$ and $v \in X$ but $u+v \notin X$ because,

$$\left(2 + e^{\frac{2\pi}{3}i}\right)^3 = 3\sqrt{3}i,$$

$$\left(2e^{\frac{2\pi}{3}i} + 1\right)^3 = -3\sqrt{3}i.$$

□

1.29. Exercise 1.C.7: A Quasi-subspace Unclosed under Scalar Multiplication

Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbb{R}^2$.

ANSWER.

$$U = \{(0,0), (1,1), (-1,-1)\}.$$

□

1.30. Exercise 1.C.8: A Quasi-subspace Unclosed under Vector Addition

Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

ANSWER.

$$U = \{(0,y) : y \in \mathbb{R}\} \cup \{(x,0) : x \in \mathbb{R}\}.$$

□

1.31. Exercise 1.C.9: The Set of Periodic Functions from $\mathbb{R}$ to $\mathbb{R}$ is NOT a Subspace of $\mathbb{R}^\mathbb{R}$

A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^\mathbb{R}$? Explain.

LEMMA 1.1. $f(x) = \cos(\pi x) + \cos(x)$ *is not periodic.*

PROOF. Suppose that $f(x) = \cos(\pi x) + \cos(x)$ is periodic, then there exists $\theta \in \mathbb{R}$ such that $\theta > 0$ and $f(x+\theta) = f(x)$ for all $x \in \mathbb{R}$. Pick $x = 0$, we have

$$f(\theta + 0) = \cos(\pi\theta) + \cos(\theta) = f(0) = \cos(0) + \cos(0) = 2. \tag{1.1}$$

Note that $-1 \leqslant \cos(\pi\theta) \leqslant 1$ and $-1 \leqslant \cos(\theta) \leqslant 1$. However, if $\cos(\pi\theta) < 1$ and $\cos(\theta) < 1$, then $\cos(\pi\theta) + \cos(\theta) < 1$. Thus (1.1) implies that

$$\begin{cases} \cos(\pi\theta) = 1 \\ \cos(\theta) = 1 \end{cases}.$$

This further implies that there exists some $m, n \in \mathbb{Z}$ such that,

$$\begin{cases} \pi\theta = 2m\pi \\ \theta = 2n\pi \end{cases}.$$

Since $\theta \neq 0$, dividing the above two equations yields

$$\pi = \frac{m}{n}.$$

This contradicts with the fact that π is irrational. □

MAIN ANSWER. No, because Lemma 1.1 gives an example of two periodic functions where their addition is not periodic. □

1.32. Exercise 1.C.10: The Intersection of 2 Subspaces is a Subspace

Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

PROOF. We prove by verifying the three conditions for $U_1 \cap U_2$ being a subspace.

(1) The premise that U_1 is a subspace implies $0 \in U_1$. The premise that U_2 is a subspace implies $0 \in U_2$. Therefore $0 \in U_1 \cap U_2$.
(2) Pick $u, w \in U_1 \cap U_2$. The premise that U_1 is a subspace implies $u + w \in U_1$. The premise that U_2 is a subspace implies $u + w \in U_2$. Therefore $u + w \in U_1 \cap U_2$.
(3) Pick $a \in \mathbb{F}$ and $u \in U_1 \cap U_2$. The premise that U_1 is a subspace implies that $au \in U_1$. The premise that U_2 is a subspace implies $au \in U_2$. Therefore $au \in U_1 \cap U_2$.

□

1.33. Exercise 1.C.11: The Intersection of Every Collection of Subspaces is a Subspace

Prove that the intersection of every collection of subspaces of V is a subspace of V.

PROOF. Pick set I and $(S_i)_{i \in I}$, a family of sets indexed over set I. Suppose that S_i is a subspace of V for all $i \in I$. We prove by verifying the three conditions for $\cap_{i \in I} S_i$ being a subspace.

(1) For all $i \in I$, the premise that S_i is a subspace implies that $0 \in S_i$. Therefore $0 \in \cap_{i \in I} S_i$.
(2) Pick $u, w \in \cap_{i \in I} S_i$. For all $i \in I$, the premise that S_i is a subspace implies $u + w \in S_i$. Therefore $u + w \in \cap_{i \in I} S_i$.
(3) Pick $a \in \mathbb{F}$ and $u \in \cap_{i \in I} S_i$. For all $i \in I$, the premise that U_i is a subspace implies that $au \in U_i$. Therefore $au \in \cap_{i \in I} S_i$.

□

1.34. Exercise 1.C.12: When the Union of 2 Subspaces is a Subspace

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

PROOF. Pick two subspaces V_1, V_2 of V.

(1) If $V_1 \subset V_2$, then $V_1 \cup V_2 = V_2$. Since V_2 is a subspace, it follows that $V_1 \cup V_2$ is a subspace.
(2) Suppose that $V_1 \cup V_2$ is a subspace. To prove by contradiction, suppose that $V_1 \not\subset V_2$ and $V_2 \not\subset V_1$. Thus there exists $v_1 \in V_1$ such that $v_1 \notin V_2$, and there exists $v_2 \in V_2$ such that $v_2 \notin V_1$.

    We claim that $v_1 + v_2 \notin V_1$. Because if $v_1 + v_2$ was in V_1, the premise that V_1 is a subspace would imply that $v_2 = (v_1 + v_2) - v_1$ is also in V_1.

    We claim that $v_1 + v_2 \notin V_2$. Because if $v_1 + v_2$ was in V_2, the premise that V_2 is a subspace would imply that $v_1 = (v_1 + v_2) - v_2$ is also in V_2.

    Thus $v_1 + v_2 \notin V_1 \cup V_2$, contradicting with the premise that $V_1 \cup V_2$ is a subspace.

□

1.35. Exercise 1.C.13: When the Union of 3 Subspaces is a Subspace

Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

PROOF. Pick three subspaces V_1, V_2, V_3 of V.

(1) If $V_1 \subset V_3$ and $V_2 \subset V_3$, then $V_1 \cup V_2 \cup V_3 = V_3$. Since V_3 is a subspace, it follows that $V_1 \cup V_2 \cup V_3$ is a subspace.

(2) Suppose that $V_1 \cup V_2 \cup V_3$ is a subspace of V.

If $V_1 \subset V_2$, then $V_1 \cup V_2 \cup V_3$ becomes $V_2 \cup V_3$. By Exercise 1.C.12, it follows that either $V_2 \subset V_3$ or $V_3 \subset V_2$. Thus either V_3 or V_2 is the subspace that contains the other two.

If $V_2 \subset V_1$, then $V_1 \cup V_2 \cup V_3$ becomes $V_1 \cup V_3$. By Exercise 1.C.12, it follows that either $V_1 \subset V_3$ or $V_3 \subset V_1$. Thus either V_3 or V_1 is the subspace that contains the other two.

If $V_1 \not\subset V_2$ and $V_2 \not\subset V_1$, then there exists $u \in V_1$ and $w \in V_2$ such that $u \notin V_2$ and $w \notin V_1$.

(i) We claim that $V_1 \setminus (V_1 \cap V_2) \subset V_3$. Pick $x \in V_1 \setminus (V_1 \cap V_2) \subset V_3$. The premise that $V_1 \cup V_2 \cup V_3$ is a subspace implies that $x+w \in V_1 \cup V_2 \cup V_3$ and $2x+w \in V_1 \cup V_2 \cup V_3$. If $x+w$ was in V_1, then $w = (x+w) - x$ would be in V_1 which is a contradiction. If $2x+w$ was in V_1, then $w = (2x+w) - 2x$ would be in V_1 which is a contradiction. If $x+w$ was in V_2, then $x = (x+w) - w$ would be in V_2 which is a contradiction. If $2x+w$ was in V_2, then $2x = (2x+w) - w$ would be in V_2 which is a contradiction. Therefore $x+w \in V_3$ and $2x+w \in V_3$. And it follows that $x = (2x+w) - (x+w)$ is in V_3.

(ii) We claim that $V_2 \setminus (V_1 \cap V_2) \subset V_3$. Pick $y \in V_2 \setminus (V_1 \cap V_2) \subset V_3$. The premise that $V_1 \cup V_2 \cup V_3$ is a subspace implies that $u+y \in V_1 \cup V_2 \cup V_3$ and $u+2y \in V_1 \cup V_2 \cup V_3$. If $u+y$ was in V_1, then $y = (u+y) - u$ would be in V_1 which is a contradiction. If $u+2y$ was in V_1, then $2y = (u+2y) - u$ would be in V_1 which is a contradiction. If $u+y$ was in V_2, then $u = (u+y) - y$ would be in V_2 which is a contradiction. If $u+2y$ was in V_2, then $u = (u+2y) - 2y$ would be in V_2 which is a contradiction. Therefore $u+y \in V_3$ and $u+2y \in V_3$. And it follows that $y = (u+2y) - (u+y)$ is in V_3.

(iii) We claim that $V_1 \cap V_2 \subset V_3$. Pick $z \in V_1 \cap V_2$. The premise that V_1 is a subspace implies that $u+z \in V_1$. If $u+z \in V_1 \cap V_2$, then $u = (u+z) - z \in V_2$ which is a contradiction. Thus $u+z \in V_1 \setminus (V_1 \cap V_2)$. By (i) we have $u+z \in V_3$. Also by (i) we have $u \in V_3$. The premise that V_3 is a subspace implies that $z = (u+z) - u$ is in V_3.

The above (i) and (iii) implies that $V_1 \subset V_3$. The above (ii) and (iii) implies that $V_2 \subset V_3$.

□

1.36. Exercise 1.C.14: An Example of Sums of Subspaces

Verify the assertion in Example 1.38: Suppose that $U = \{(x,x,y,y) \in \mathbb{F}^4 : x,y \in \mathbb{F}\}$ and W=$\{(x,x,x,y) \in \mathbb{F}^4 : x,y \in \mathbb{F}\}$. Then

$$U + W = \{(x,x,y,z) \in \mathbb{F}^4 : x,y,z \in \mathbb{F}\}.$$

PROOF. (1) Pick $x \in U + W$. Then there exists $u = (x_u, x_u, y_u, y_u) \in U$ and $w = (x_w, x_w, x_w, y_w) \in W$ such that $x_u, y_u, x_w, y_w \in \mathbb{F}$ and $x = u + w$. Thus $x = (x_u + x_w, x_u + x_w, y_u + x_w, y_u + y_w)$, which shows that $x \in \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\}$. Therefore $U + W \subset \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\}$.

(2) Pick $v \in \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\}$. Then there exists $x, y, z \in \mathbb{F}$ such that $v = (x, x, y, z)$. By choosing $u = (x, x, y, y) \in U$ and $w = (0, 0, 0, z - y) \in W$, we have $u + w = v$. Thus $v \in U + W$. Therefore $\{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\} \subset U + W$.

The above (1) and (2) imply that $U + W = \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\}$. □

1.37. Exercise 1.C.15: $U + U = U$ for Any Subspace U

Suppose U is a subspace of V. What is $U + U$?

PROOF. We claim that $U + U = U$.

(1) Pick $z \in U + U$, then there exists $x, y \in U$ such that $z = x + y$. Since U is closed under addition, we have $x + y \in U$, i.e. $z \in U$. Thus $U + U \subset U$.

(2) Pick $u \in U$, then $u = 1u = (\frac{1}{2} + \frac{1}{2})u = \frac{1}{2}u + \frac{1}{2}u$, i.e. $u \in U + U$. Thus $U \subset U + U$.

The above (1) and (2) imply that $U + U = U$. □

1.38. Exercise 1.C.16: Commutativity of the Sum of Subspaces

Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

PROOF. We claim that $U + W = W + U$.

(1) Pick $v \in U + W$, then there exists $u \in U$ and $w \in W$ such that $v = u + w$. Since the addition on V is commutative, $u + w = w + u$, and this shows that $v = w + u$ is in $W + U$. Thus $U + W \subset W + U$.

(2) Pick $v \in W + U$, then there exists $w \in W$ and $u \in U$ such that $v = w + u$. Since the addition on V is commutative, $w + u = u + w$, and this shows that $v = u + w$ is in $U + W$. Thus $W + U \subset U + W$.

The above (1) and (2) imply that $U + W = W + U$. □

1.39. Exercise 1.C.17: Associativity of the Sum of Subspaces

Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, U_3 are subspaces of V, is

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

PROOF. We claim that $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$.

(1) Pick $v \in (U_1 + U_2) + U_3$, then there exists $u_1, u_2, u_3 \in U$ such that $v = (u_1 + u_2) + u_3$. Since the addition on V is associative, $(u_1 + u_2) + u_3 = u_1 + (u_2 + u_3)$, and this shows that $v = u_1 + (u_2 + u_3)$ is in $U_1 + (U_2 + U_3)$. Thus $(U_1 + U_2) + U_3 \subset U_1 + (U_2 + U_3)$.

(2) Pick $v \in U_1 + (U_2 + U_3)$, then there exists $u_1, u_2, u_3 \in U$ such that $v = u_1 + (u_2 + u_3)$. Since the addition on V is associative, $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$, and this shows that $v = (u_1 + u_2) + u_3$ is in $(U_1 + U_2) + U_3$. Thus $U_1 + (U_2 + U_3) \subset (U_1 + U_2) + U_3$.

The above (1) and (2) imply that $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$. □

1.40. Exercise 1.C.18: The Identity of Sum of Subspaces

Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

PROOF. We claim that the subspace $\{0\}$ is the additive identity, i.e. $U + \{0\} = U$ for all subspaces U.

(1) Pick $v \in U + \{0\}$, then there exists $u \in U$ such that $v = u + 0$, and this shows that $v = u$ is in U. Thus $U + \{0\} \subset U$.
(2) Pick $u \in U$, then $u = u + 0$ shows that $u \in U + \{0\}$. Thus $U \subset U + \{0\}$.

The above (1) and (2) imply that $U + \{0\} = U$.

We claim that for all subspaces U and W, $U + W = \{0\}$ implies $U = W = \{0\}$, which means that the subspaces that have additive inverses are only $\{0\}$ itself.

Pick subspaces U and W. Assume $U + W = \{0\}$. 1.39 of [**Axl14**] shows that $U \subset U + W$ and $W \subset U + W$. Thus $U \subset \{0\}$ and $W \subset \{0\}$. Note that the only two subsets of $\{0\}$ are $\varnothing$ and $\{0\}$. Also, the premise that U and W are subspaces implies $0 \in U$ and $0 \in W$, thus $U = W = \{0\}$. □

1.41. Exercise 1.C.19: There is NO Cancellation Law for Sums of Subspaces

Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

COUNTEREXAMPLE. Set $V = \mathbb{R}^2, U_1 = \mathbb{R}^2, U_2 = \{(x, 0) : x \in \mathbb{R}\}, W = \{(0, x) : x \in \mathbb{R}\}$. Then U_1, U_2, W are subspaces of V, $U_1 + W = \mathbb{R}^2$, $U_2 + W = \mathbb{R}^2$ but $U_1 \neq U_2$. □

1.42. Exercise 1.C.20: A Direct Sum of 2 Subspaces that Equals to $\mathbb{F}^4$

Suppose

$$U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

PROOF. We claim that the following W is a subspace of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$:

$$W = \{(0, x, 0, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}.$$

We first verify that W is a subspace of $\mathbb{F}^4$ by 1.34 of [**Axl14**].

(1) We can show that $0 \in W$ by choosing $x = 0, y = 0$ in $(0, x, 0, y)$.
(2) Suppose that $u, w \in W$ where $u = (0, x_u, 0, y_u)$ and w=$(0, x_w, 0, y_w)$. Then $u + w = (0, x_u + x_w, 0, y_u + y_w)$ shows that $u + w$ is in W.
(3) Pick $a \in \mathbb{F}$ and $u \in W$ where $u = (0, x_u, 0, y_u)$. Then $au = (0, ax_u, 0, ay_u)$ shows that au is in W.

We then verify that $U + W$ is a direct sum by 1.45 of [**Axl14**].

Pick $v \in U \cap W$. Then there exists $x, y, s, t \in \mathbb{F}$ such that $v = (x, x, y, y) = (0, s, 0, t)$, which implies that $x = 0, s = 0, y = 0, t = 0$, i.e. $v = 0$. Therefore $U + W$ is a direct sum.

Finally, we verify that $\mathbb{F}^4 = U \oplus W$.

(1) 1.39 of [**Axl14**] implies that $U \oplus W$ is a subspace of $\mathbb{F}^4$. Therefore $U \oplus W \subset \mathbb{F}^4$.
(2) Pick $v \in \mathbb{F}^4$ where $v = (a, b, c, d)$ and $a, b, c, d \in \mathbb{F}$. Define

$$u = (a, a, c, c), w = (0, b - a, 0, d - c).$$

Then $u \in U, w \in W$ and $v = u + w$, which shows that $v \in U \oplus W$. Thus $\mathbb{F}^4 \subset U \oplus W$.

□

1.43. Exercise 1.C.21: A Direct Sum of 2 Subspaces that Equals to $\mathbb{F}^5$

Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F} = U \oplus W$.

PROOF. We claim that the following W is a subspace of $\mathbb{F}^5$ such that $\mathbb{F}^4 = U \oplus W$:

$$W = \{(0, 0, x, y, z) \in \mathbb{F}^5 : x, y, z \in \mathbb{F}\}.$$

We first verify that W is a subspace of $\mathbb{F}^5$ by 1.34 of [**Axl14**].

(1) We can show that $0 \in W$ by choosing $x = 0, y = 0, z = 0$ in $(0, 0, x, y, z)$.
(2) Suppose that $u, w \in W$ where

$$u = (0, 0, x_u, y_u, z_u), w = (0, 0, x_w, y_w, z_w).$$

Then $u + w = (0, 0, x_u + x_w, y_u + y_w, z_u + z_w)$ shows that $u + w$ is in W.
(3) Pick $a \in \mathbb{F}$ and $u \in W$ where $u = (0, 0, x_u, y_u, z_u)$. Then

$$au = (0, 0, ax_u, ay_u, az_u)$$

shows that au is in W.

We then verify that $U + W$ is a direct sum by 1.45 of [**Axl14**].

Pick $v \in U \cap W$. Then there exists $x, y, a, b, c \in \mathbb{F}$ such that $v = (x, y, x + y, x - y, 2x) = (0, 0, a, b, c)$, which implies that $x = 0, y = 0, a = 0, b = 0, c = 0$, i.e. $v = 0$. Therefore $U + W$ is a direct sum.

Finally, we verify that $\mathbb{F}^5 = U \oplus W$.

(1) 1.39 of [**Axl14**] implies that $U \oplus W$ is a subspace of $\mathbb{F}^5$. Therefore $U + W \subset \mathbb{F}^5$.
(2) Pick $v \in \mathbb{F}^5$ where $v = (a, b, c, d, e)$ and $a, b, c, d, e \in \mathbb{F}$. Define $u = (a, b, a+b, a-b, 2a), w = (0, 0, c-a-b, d-a+b, e-2a)$. Then $u \in U, w \in W$ and $v = u + w$, which shows that $v \in U + W$. Thus $\mathbb{F}^5 \subset U \oplus W$.

□

1.44. Exercise 1.C.22: A Direct Sum of 4 Subspaces that Equals to $\mathbb{F}^5$

Suppose

$$U = \{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

PROOF. We claim that the following W_1, W_2, W_3 are subspaces of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$:

$$W_1 = \{(0,0,x,0,0) \in \mathbb{F}^5 : x \in \mathbb{F}\},$$
$$W_2 = \{(0,0,0,x,0) \in \mathbb{F}^5 : x \in \mathbb{F}\},$$
$$W_3 = \{(0,0,0,0,x) \in \mathbb{F}^5 : x \in \mathbb{F}\}.$$

Suppose that

$$0 = u + w_1 + w_2 + w_3 \tag{1.2}$$

for some $u = (x, y, x+y, x-y, 2x), w_1 = (0,0,a,0,0), w_2 = (0,0,0,b,0), w_3 = (0,0,0,0,c)$ where $x, y, a, b, c \in \mathbb{F}$. Then (1.2) is equivalent to the following system of linear equations.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is now clear that the coefficient matrix is nonsingular, thus $x = y = a = b = c = 0$ is the only solution. By 1.44 of [**Axl14**], we conclude that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$. □

1.45. Exercise 1.C.23: There is NO Cancellation Law for Direct Sums of Subspaces

Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W,$$

then $U_1 = U_2$.

COUNTEREXAMPLE. Set $V = \mathbb{R}^2, U_1 = \{(x,x) : x \in \mathbb{R}\}, U_2 = \{(-x,x) : x \in \mathbb{R}\}, W = \{(0,x) : x \in \mathbb{R}\}$. Then U_1, U_2, W are subspaces of V, $U_1 \oplus W = \mathbb{R}^2$, $U_2 \oplus W = \mathbb{R}^2$ but $U_1 \neq U_2$. □

1.46. Exercise 1.C.24: A Direct Sum Decomposition of $\mathbb{R}^{\mathbb{R}}$

A function $f : \mathbb{R} \to \mathbb{R}$ is called even if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called odd if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

PROOF. We first verify that U_e and U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$ by 1.34 of [**Axl14**].

(1) We can show that $0 \in U_e$ and $0 \in U_o$ because the $0(-x) = 0(x) = 0$ and $0(-x) = -0(x) = 0$ for all $x \in \mathbb{R}$.
(2) Suppose that $u_e, w_e \in U_e, u_o, w_o \in U_o$, then

$$\begin{aligned}(u_e + w_e)(-x) &= u_e(-x) + w_e(-x)\\ &= u_e(x) + w_e(x)\\ &= (u_e + w_e)(x).\end{aligned}$$

Thus $u_e + w_e \in U_e$. Also,

$$\begin{aligned}(u_o + w_o)(-x) &= u_o(-x) + w_o(-x)\\ &= -u_o(x) - w_o(x)\\ &= -(u_o(x) + w_o(x))\\ &= -(u_o + w_o)(x).\end{aligned}$$

Thus $u_o + w_o \in U_o$.
(3) Pick $a \in \mathbb{F}$ and $u_e \in U_e, u_o \in U_o$, then

$$\begin{aligned}(au_e)(-x) &= a \cdot u_e(-x)\\ &= a \cdot u_e(x)\\ &= (au_e)(x).\end{aligned}$$

Thus $au_e \in U_e$. Also,

$$\begin{aligned}(au_o)(-x) &= a \cdot u_o(-x)\\ &= -a \cdot u_o(x)\\ &= -(au_o)(x).\end{aligned}$$

Thus $au_o \in U_o$.

We then verify that $U_e + U_o$ is a direct sum by 1.45 of [**Axl14**].

Pick $u \in U_e \cap U_o$, then $u(-x) = u(x) = -u(x)$ for all $x \in \mathbb{R}$. Thus $u(x) = 0$ for all $x \in \mathbb{R}$, which means that u is the 0 function.

Finally, we prove that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

(1) 1.39 of [**Axl14**] implies that $U_e \oplus U_o$ is a subspace of $\mathbb{R}^{\mathbb{R}}$. Therefore $U_e \oplus U_o \subset \mathbb{R}^{\mathbb{R}}$.
(2) Pick $f(x) \in \mathbb{R}^{\mathbb{R}}$. Define

$$f_e(x) = \frac{f(x) + f(-x)}{2}, f_o(x) = \frac{f(x) - f(-x)}{2}.$$

Then

$$\begin{aligned}f_e(-x) &= \frac{f(-x) + f(x)}{2}\\ &= \frac{f(x) + f(-x)}{2}\\ &= f_e(x),\end{aligned}$$

which implies that $f_e(x) \in U_e$. Also,

$$\begin{aligned} f_o(-x) &= \frac{f(-x) - f(x)}{2} \\ &= -\frac{f(x) - f(-x)}{2} \\ &= -f_o(x), \end{aligned}$$

which implies that $f_o(x) \in U_o$. And,

$$\begin{aligned} f_e(x) + f_o(x) &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\ &= -\frac{f(x) + f(-x) + f(x) - f(-x)}{2} \\ &= f(x), \end{aligned}$$

which shows that $f(x) \in U_e \oplus U_o$. Therefore $\mathbb{R}^{\mathbb{R}} \subset U_e \oplus U_o$.

□

CHAPTER 2

Finite-Dimensional Vector Space

2.1. Exercise 2.A.1

Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

PROOF. Denote $w_1 = v_1 - v_2, w_2 = v_2 - v_3, w_3 = v_3 - v_4, w_4 = v_4$. Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then

$$A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = A^{-1} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}.$$

Moreover, it can be calculated that

$$A^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(1) Pick $x \in \operatorname{span}(v_1, v_2, v_3, v_4)$, then there exists $x_1, x_2, x_3, x_4 \in \mathbb{F}$ such that

$$\begin{aligned} v &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_1 + x_2 & x_1 + x_2 + x_3 & x_1 + x_2 + x_3 + x_4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}. \end{aligned}$$

Therefore $v \in \operatorname{span}(w_1, w_2, w_3, w_4)$.

(2) Pick $w \in \operatorname{span}(w_1, w_2, w_3, w_4)$, then there exists $y_1, y_2, y_3, y_4 \in \mathbb{F}$ such that

$$\begin{aligned} w &= \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \\ &= \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \\ &= \begin{bmatrix} y_1 & y_2 - y_1 & y_3 - y_2 & y_4 - y_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}. \end{aligned}$$

Therefore $w \in \operatorname{span}(v_1, v_2, v_3, v_4)$.

The above (1) and (2) imply that $\operatorname{span}(w_1, w_2, w_3, w_4) = \operatorname{span}(v_1, v_2, v_3, v_4)$. □

2.2. Exercise 2.A.2

Verify the assertions in Example 2.18.

(a) A list v of one vector $v \in V$ is linearly independent if and only if $v \neq 0$.
(b) A list of two vectors in V is linearly independent if and only if neither vector is a scalar multiple of the other.
(c) $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ is linearly independent in $\mathbb{F}^4$.
(d) The list $1, z, \ldots, z^m$ is linearly independent in $\mathcal{P}(\mathbb{F})$ for each nonnegative integer m.

Proof. (a) If the list v is linearly independent but $v = 0$, then $1v = 0$ is a nontrivial representation of 0 using v which is a contradiction. If $v \neq 0$ but the list v is not linearly independent, then there exists $a \in \mathbb{F}$ such that $av = 0$ and $a \neq 0$ which contradicts 1.30 of [**Axl14**].

(b) If a list of two vectors v_1, v_2 in V is linearly independent and assume that, without loss of generality, $v_1 = av_2$ for some $a \in \mathbb{F}$, then $1v_1 - av_2 = 0$ is a nontrivial representation of 0 using v_1, v_2 which is a contradiction. If a list of two vectors v_1, v_2 in V is not linearly independent, then there exist some $a_1, a_2 \in \mathbb{F}$ such that $a_1v_1 + a_2v_2 = 0$ and not both of a_1 and a_2 are zero. If $a_1 \neq 0$, then $v_1 = -\frac{a_2}{a_1}v_2$; if $a_1 = 0$, then $v_2 = -\frac{a_1}{a_2}v_1$.

(c) Consider the equation $x(1, 0, 0, 0) + y(0, 1, 0, 0) + z(0, 0, 1, 0) = 0$ where $x, y, z \in \mathbb{F}$. In matrix form,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the coefficient matrix is nonsingular, the only solution is $x = y = z = 0$.

(d) Consider the function $f(z) = x_0 + x_1z + \cdots + x_{N-1}z^{N-1}$ where $N - 1 \geq 0$ and $x_0, x_1, \ldots, x_{N-1} \in \mathbb{F}$. Suppose $f(z) = 0$ for all $z \in \mathbb{F}$. Then, in particular, $f(1) = 0, f(2) = 0, \ldots, f(N) = 0$. In matrix form,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & 2^3 & \cdots & 2^{N-1} \\ 1 & 3 & 3^2 & 3^3 & \cdots & 3^{N-1} \\ 1 & 4 & 4^2 & 4^3 & \cdots & 4^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & N & N^2 & N^3 & \cdots & N^{N-1} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Note that the above coefficient matrix is a Vandermonde matrix. Since all the numbers $1, 2, \ldots, N$ are distinct, its determinant is non-zero. Therefore the only solution is $x_0 = x_1 = \cdots = x_{N-1} = 0$.

□

2.3. Exercise 2.A.3

Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbb{R}^3$.

PROOF. We claim that $t = 2$ makes the list not linearly independent in $\mathbb{R}^3$. This can be shown by

$$\begin{aligned} & -3 \cdot (3, 1, 4) + 2 \cdot (2, -3, 5) + 1 \cdot (5, 9, 2) \\ &= ((-9) + 4 + 5, (-3) + (-6) + 9, (-12) + 10 + 2) \\ &= 0. \end{aligned}$$

□

2.4. Exercise 2.A.4

Verify the assertion in the second bullet point in Example 2.20: the list

$$(2, 3, 1), (1, -1, 2), (7, 3, c)$$

is linearly dependent in $\mathbb{F}^3$ if and only if $c = 8$.

PROOF. Let

$$A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & -1 & 3 \\ 1 & 2 & c \end{bmatrix}.$$

Then the list $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent if and only if $\det(A) = 0$. Since $\det(A) = 40 - 5c$, it follows that $\det(A) = 0$ if and only if $c = 8$. □

2.5. Exercise 2.A.5

(a) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list $(1+i, 1-i)$ is linearly independent.
(b) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the list $(1+i, 1-i)$ is linearly dependent.

PROOF. Suppose that there exists $a+bi, c+di \in \mathbb{C}$ such that $a,b,c,d \in \mathbb{R}$ and $(a+bi)(1+i)+(c+di)(1-i)=0$, then

$$(a-b+c+d)+(a+b-c+d)i=0,$$

Rewrite the above linear system in the matrix form,

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then its solutions are

$$\left\{ \begin{bmatrix} -y \\ x \\ x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

(a) If we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list is linearly independent because

$$\left\{ \begin{bmatrix} -y \\ x \\ x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\} \cap \left\{ \begin{bmatrix} x \\ 0 \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(b) If we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the list is linearly dependent because

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \left\{ \begin{bmatrix} -y \\ x \\ x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

.

□

2.6. Exercise 2.A.6

Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

PROOF. Use the same notation as the proof of Exercise 2.A.1. If the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is linearly dependent, then there exists $a_1, a_2, a_3, a_4 \in \mathbb{F}$, such that

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0.$$

Since A is nonsingular,

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} A \neq \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore it follows that the list v_1, v_2, v_3, v_4 is linearly dependent. □

2.7. Exercise 2.A.7

Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

is linearly independent.

Proof. We claim that the proposition is true. Define

$$A = \begin{bmatrix} 5 & -4 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Suppose that

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

is linearly dependent, then there exists $a_1, a_2, a_3, \ldots, a_m \in \mathbb{F}$, such that

$$\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_m \end{bmatrix} A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \ldots \\ v_m \end{bmatrix} = 0$$

and

$$\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_m \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Since $\det(A) = 5$, the matrix A is nonsingular. Thus

$$\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_m \end{bmatrix} A \neq \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Therefore it follows that the list $v_1, v_2, \ldots, v_m$ is linearly dependent. □

2.8. Exercise 2.A.8

Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

Proof. We claim that the proposition is true. Suppose that $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly dependent, then there exists $a_1, a_2, \ldots, a_m$, not all 0, such that

$$a_1 \lambda v_1 + a_2 \lambda v_2 + \cdots + a_m \lambda v_m = 0.$$

Since $\lambda \neq 0$, λ^{-1} exists in $\mathbb{F}$. Multiplying the above equation by λ^{-1} yields

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0.$$

Note that not all of $a_1, a_2, \ldots, a_m$ are zero. Therefore it follows that $v_1, v_2, \ldots, v_m$ is a linearly dependent list. □

2.9. Exercise 2.A.9

Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

COUNTEREXAMPLE. Set $V = \mathbb{R}^2, m = 2, v_1 = (0,1), v_2 = (1,0), w_1 = (0,-1), w_2 = (-1,0)$. Then v_1, v_2 is a linearly independent list, and w_1, w_2 is also a linearly independent list. But

$$v_1 + w_1 = (0,0)$$

and

$$v_2 + w_2 = (0,0)$$

is not a linearly independent list. □

2.10. Exercise 2.A.10

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

PROOF. If $v_1+w, \ldots, v_m+w$ is linearly dependent, then there exists $a_1, \ldots, a_m$, not all 0, such that

$$a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0.$$

The above equation implies that

$$w = -\frac{a_1}{a_1 + \cdots + a_m} v_1 - \cdots - \frac{a_m}{a_1 + \cdots + a_m} v_m,$$

thus $w \in \operatorname{span}(v_1, \ldots, v_m)$. □

Bibliography

[Axl14] S. Axler, *Linear algebra done right*, Undergraduate Texts in Mathematics, Springer International Publishing, 2014.

[Rud76] W. Rudin, *Principles of mathematical analysis*, International series in pure and applied mathematics, McGraw-Hill, 1976.