Minqi's Solutions to the Exercises of Walter Rudin (1976), Principles of Mathematical Analysis (3rd Edition)

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Contents

CHAPTER 1

The Real and Complex Number Systems

1.1. Exercise 1: Mixed Operations between Rational and Irrational Numbers Produce Irrational Numbers

If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

PROOF. Denote $\alpha = r + x, \beta = rx$.

Suppose $\alpha \in \mathbb{Q}$. Then $x = \alpha - r$ is also in $\mathbb{Q}$, because $\mathbb{Q}$ is a field that is closed with respect to additive inverse and addition. This contradicts with the presumption that $x \notin \mathbb{Q}$.

Suppose $\beta \in \mathbb{Q}$. Then $x = \beta/r$ is also in $\mathbb{Q}$, because $r \neq 0$ and $\mathbb{Q}$ is a field that is closed with respect to multiplicative inverse and multiplication. This contradicts with the presumption that $x \notin \mathbb{Q}$. □

1.2. Exercise 2: $\sqrt{12} \notin \mathbb{Q}$

Prove that there is no rational number whose square is 12.

LEMMA 1.1. $\forall n \in \mathbb{Z}, \forall p \in \{2,3\}$,

$$n^2 \equiv 0 \implies n \equiv 0 \pmod{p}.$$

PROOF. We check the squares of every element of $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$, the ring of integers modulo 2.

$$0^2 = 0$$
$$1^2 = 1$$

Thus $n^2 = 0$ implies $n = 0$ in $\mathbb{Z}/2\mathbb{Z}$ by exhaustion.

Similarly we check the squares of every element in $\mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$, the ring of integers modulo 3.

$$0^2 = 0$$
$$1^2 = 1$$
$$2^2 = 1$$

Thus $n^2 = 0$ implies $n = 0$ in $\mathbb{Z}/3\mathbb{Z}$ by exhaustion. □

LEMMA 1.2. *There is no rational number whose square is* 3.

PROOF. We now show that the equation

$$p^2 = 3 \tag{1.1}$$

is not satisfied by any rational p. If there was such a p, we could write $p = \frac{m}{n}$ where $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 3$ are not both true. Let us assume this is done. Then 1.1 implies

$$m^2 = 3n^2, \tag{1.2}$$

This shows that $m^2 \equiv 0 \pmod 3$ and we have $m \equiv 0 \pmod 3$ by Lemma 1.1. So $m^2 \equiv 0 \pmod 9$. It follows that the right side of 1.2 is divisible by 9, so that $n^2 \equiv 0 \pmod 3$, which implies that $n \equiv 0 \pmod 3$ again by Lemma 1.1.

The assumption that 1.1 holds thus leads to the conclusion that both $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 3$ are true, contrary to our choice of m and n. Hence 1.1 is impossible for rational p. □

Main Proof. Suppose that there existed $p \in \mathbb{Q}$ such that $p^2 = 12$. Then there exists $m, n \in \mathbb{Z}$ such that

$$p = \frac{m}{n} \implies 12 = \frac{m^2}{n^2} \implies m^2 = 2 \times 6n^2 \implies m^2 \equiv 0 \pmod 2.$$

So it follows from Lemma 1.1 that

$$m \equiv 0 \pmod 2.$$

Therefore there exists $k \in \mathbb{Z}$ such that $m = 2k$, and

$$4k^2 = 12n^2 \implies k^2 = 3n^2 \implies \left(\frac{k}{n}\right)^2 = 3.$$

This contradicts with Lemma 1.2.

□