$\sqrt{2}$ IS NOT A RATIONAL NUMBER

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Lemma. *$\forall n \in \mathbb{N}$, n^2 is even implies that n is even.*

Proof. Suppose that n is odd, then

$$n = 2k + 1$$

for some $k \in \mathbb{N}$. Thus,

$$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

which means n^2 is odd. The lemma thus holds by contraposition. □

Proposition. *$\sqrt{2}$ is not a rational number.*

Proof. Suppose that $\sqrt{2}$ is a rational number, then

$$\sqrt{2} = \frac{m}{n}$$

for some $m \in \mathbb{N}$, $n \in \mathbb{N}$ and (m, n) are co-prime integers. Also,

$$2 = \frac{m^2}{n^2}$$

which means that

$$m^2 = 2n^2 \tag{1}$$

thus m^2 is even. And by Lemma, m is even. Thus there exists $k \in \mathbb{N}$ such that

$$m = 2k \tag{2}$$

Now substitute equation 2 into equation 1,

$$4k^2 = 2n^2$$

That is,

$$2k^2 = n^2$$

Thus n^2 is even. By Lemma, n is even. Thus n and m are both even, contradicting the proposition that (m, n) are co-prime integers. □

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